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Coupled electron-phonon system with Coulomb interaction

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In this paper quasiparticle spectra of an electron-phonon-Coulomb system have been discussed Explicit expressions for renormalized phonon frequency, phonon attenuation and electron self energy have been derived. Spectral weight function for such system has been graphically evaluated. It has also been shown that on the basis of the quasiparticle picture, the phonon self-energy is much greater than the electron self-energy. Finally, starting from the Bardeen-Pines Hamiltonian the results have been extended to include the effects of ionic motion.

1. Introduction

Migdal (1958) was able to solve the self-energy integral by employing the approximation that the so-called 3-point electron-phonon vertex function can be replaced by unity at the cost of an error of the order of the square root of the electron to ion mass ratio $(m/M)^{\frac{1}{2}}$ which is negligible for normal metals. However, in his treatment the effect of the Coulomb interaction was not taken into account. Later, Engelsberg & Schrieffer (1963) studied the quasiparticle spectra of a coupled electron-Phonon system on the basis of Einstein and Debye models. In this paper also the Coulomb interaction was not included In recent literature, the problem of Coulomb interaction has been discussed with refeence to various properties of solids. The notable contributions are found in the papers of Batyev & Pokrovskii (1964), Heine *et al* (1966), Sham (1967), Garland (1967), Prange & Sachs (1967), Horner (1971) and others. The purpose of the present paper is to derive explicit expressions for renormalized phonon frequency, phonon attenuation and electron self-energy for electron-phonon-coulomb system and to deal with the problem of inclusion of the effects of ionic motion on the basis Bardeen-Pines Hamiltonian. Spectral weight function $A(c_k, \omega_q)$ of an electron-phonon-Coulomb system has also been graphically examined. A peculiar and interesting result has been obtained in Section 7. There it has been shown on the basis of simple arguments that near the Fermi surface phonon self-energy is much greater than the electron self-energy. Whether such result will enable one to explain new features of solids is still to be investigated.

2. Self-energy expressions and Migdal's Theorem

The mathematical expression for electron self-energy may be written in the form :

$$\Sigma(p, \epsilon) = \frac{1}{i} \sum_q \gamma_q{}^0 \int \frac{d\omega}{2\pi} G(p+q, \epsilon+\omega) D(q, \omega)\gamma(p, \epsilon; q, \omega) \quad \ldots (2.1)$$

with usual notations. G and D are electron and phonon Green functions defined as

$$G_1{}^2 = -i < T(C_{p_1}(t_1)C_{p_2}{}^*(t_2)>$$

$$D_1{}^2 = -i < T(a_{q_1}(t_1)a_{q_2}{}^*(t_2))>$$

where T is the ordering according to real time,

$$T(A(t)B(t')) = \theta(t-t')\ A(t)\ B(t') \pm \theta(t'-t)\ B(t')\ A(t)$$

the upper (lower) sign referring to boson (fermion) operators. Using the Fourier transform

$$G(p, \epsilon) = \int dt\, G(p, t) \exp(ict)$$

and the Dyson equation

$$G(p, \epsilon) = G_0(p, \epsilon)+G_0(p, \epsilon)\Sigma(p, \epsilon)G(p, \epsilon),$$

one can easily arrive at the result

$$G(p, \epsilon)^{-1} = c_p{}^0-\epsilon-\Sigma(p, \epsilon)$$

Therefore, the equation (2.1) takes the form :

$$\Sigma(p, \epsilon) = \frac{1}{i} \sum_q \gamma_q{}^0 \int \frac{d\omega}{2\pi} \frac{D(q, \omega)\, \gamma(p, q; \epsilon, \omega)}{\epsilon^0{}_{p+q}-\epsilon-\omega-\Sigma(p+q, \epsilon+\omega)} \quad (2\ 2)$$

This shows that $\Sigma(p, \epsilon)$ is undefined at all those points where the energy-denominator vanishes. The problem associated with these singularities may be resolved if following Kadanoff (1964) we make the following two replacements

$$c^\circ{}_{p+q}-\omega-\Sigma(p+q, \epsilon+\omega) = Z$$

$$\sum_q \rightarrow \int \frac{d^3q}{(2\pi)^3}$$

where Z is a complex variable. Eqn. (2.2) is thus simplified to following soluble form :

$$\Sigma(p, \epsilon) = \frac{1}{i} \int \frac{d^3q'}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{1}{Z-\epsilon} D(q', \omega')\gamma_{q'}{}^0\gamma(p, \epsilon; q', \omega'). \quad \ldots (2.3)$$

For further simplification one should know the form of the vertex function. An approximate form is, however, obtained from Migdal's Theorem. Migdal (1958) states

$$\frac{\gamma(p, \epsilon; q, \omega)}{\gamma_q{}^0} = 1+O(M^{-\frac{1}{2}}).$$

Therefore, the expression (2.3) reduces to

$$\Sigma(p, \epsilon) = \frac{1}{i}\int \frac{d^3q'}{(2\pi)^3}\int \frac{d\omega'}{2\pi}\,\frac{1}{\bar{z}-\epsilon}\,D(q', \omega') \qquad \ldots \quad (2.4)$$

There are, however, certain exceptions to Migdal's theorem. Engelsberg & Schrieffer (1963) showed that the generalized Ward-Takahasi (1957) identity

$$\frac{\gamma(p, \epsilon; q, \omega)}{\gamma_q{}^0} = \frac{G^{-1}(p+q, \epsilon+\omega)-G^{-1}(p, \epsilon)}{\omega}$$

taken in the limit

$$\frac{\gamma(p, \epsilon; q, \omega)}{\gamma_q{}^0}\Big|_{\substack{q\to 0\\ \omega\to 0}} = 1-\frac{\partial\Sigma(p, \epsilon)}{\partial\epsilon}$$

may lead to a correction of order unity in some special cases.

3. Renormalized phonon frequency

We start from the fundamental Hamiltonian of the form :

$$H = \sum_{p,\sigma}\epsilon(\boldsymbol{p})C_{p\sigma}{}^{+}c_{p\sigma}+\sum_{\lambda,\, q<q_m}\omega^0{}_{q\lambda}a_{q\lambda}{}^{+}a_{q\lambda}+\sum_{p_1p_2p_3p_4}V_{p_1p_2p_3p_4}c_{p_1\sigma_1}{}^{+}c_{p_2\sigma_2}c_{p_3\sigma_2}c_{p_4\sigma_1}$$

$$+\sum_{p_1p_2q\sigma_1\sigma_2}\alpha^0(\boldsymbol{q})[a_{q(p_1p_2)\lambda}+a_{q(p_1p_2)\lambda}{}^{+}]c_{p_1\sigma_1}{}^{+}c \qquad \ldots \quad (3.1)$$

which is exactly similar to that considered by Prange & Sachs (1967). The expression for polarization propagator is given by

$$D^{-1}(\boldsymbol{q}.\ \omega) = \{V(\boldsymbol{q})+D_0(\boldsymbol{q}, \omega)\}^{-1}-\Pi \qquad \ldots \quad (3.2)$$

where $D_0(\boldsymbol{q}, \omega)$ is the bare phonon propagator

$$D_0(\boldsymbol{q}, \omega) = \frac{2\omega_q{}^0|\alpha^0(\boldsymbol{q})|^2}{\omega^2-\omega_q{}^{02}+i\eta} \qquad \ldots \quad (3.3)$$

and Π is the irreducible part of the phonon self-energy. In the above espression $|\alpha^0(\boldsymbol{q})|^2$ determines the interaction between electrons and phonons given by

$$|\alpha^0(\boldsymbol{q})|^2 = \frac{4\pi e^2}{q^2}\cdot\frac{\omega_q{}^0}{2} \qquad \ldots \quad (3.4)$$

for longitudinal phonons.

Substituting (3.3) into (3.2), we get

$$D(q,\omega)=\frac{\frac{4\pi e^2}{q^2}\omega^2}{\left(1-\frac{4\pi e^2}{q^2}\Pi\right)\omega^2-\omega_q{}^{02}} \qquad \ldots (3.5)$$

Now Migdal (1958) showed

$$\Pi(q,\omega)=\frac{p_0}{4\pi^2}\left[g(q/2p_0)+\pi i\frac{|\omega|}{2p_0q}\right] \qquad \ldots (3\,6)$$

where $$g(x)\simeq(1-x^2/2)$$

in the interval $0<x<1$. Therefore,

$$D(q,\omega)=\frac{\frac{4\pi e^2}{q^2}\omega^2\Big/\left\{1-\frac{p_0e^2}{\pi q^2}\left(1-\frac{q^2}{8p_0{}^2}+\pi i\frac{|\omega|}{2p_0q}\right)\right\}}{\omega^2-\omega_q{}^{02}\Big/\left\{1-\frac{p_0e^2}{\pi q^2}\left(1-\frac{q^2}{8p_0{}^2}+\pi i\frac{|\omega|}{2p_0q}\right)\right\}} \qquad \ldots (3.7)$$

Real part of the pole of $D(q,\omega)$ gives the renormalized phonon frequency

$$\omega_q{}^2=\omega_q{}^{02}\frac{1-F/2+F/4(q/2p_0)^2}{\{1-F/2+F/4(q/2p_0)^2\}^2+F^2\left\{\frac{\pi|\omega|}{2p_0q}\right\}^2}, \qquad \ldots (3\,8)$$

where F is the Fröhlich dimensionless coupling parameter given by

$$F=\frac{p_0e^2}{\pi q^2}\lesssim 1.$$

If the effects due to Coulomb interaction were not included, one would obtain

$$\omega_q{}'^2=\omega_q{}^{02}\left(1-F+F\frac{q^2}{8p_0{}^2}\right) \qquad \ldots (3.9)$$

For a comparative study between the eqns (3.8) and (3 9) we approximate the complicated expression (3 8) to a more simplified form. The Eqn (3.6) was obtained by Migdal by assuming the fact that the values of $\pi(q,\omega)$ which are important are obtainable for

$$\omega\sim\omega_q\sim p_0q/\sqrt{M}\ll p_0q.$$

Therefore the second term in the denominator of (3.8) becomes

$$\sim 2.5F^2M^{-1}$$

which is evidently very small. Hence (3.8) can be written approximately in the form :

$$\omega_q{}^2\simeq\omega_q{}^{02}(1-F/2+F\,q^2/16p_0{}^2)^{-1}\simeq\omega_q{}^{02}(1+F/2). \qquad \ldots (3.10)$$

Therefore,

$$\frac{\omega_q'^2}{\omega_q^2} \simeq (1-F/2)(1-F+F\,q^2/8p_0^2) \simeq 1-\frac{3}{2}F+F\frac{q^2}{8p_0^2}$$

since $(q/2p_0) < 1$, the above equation shows that the right handside is less than unity. This means $\omega_q > \omega_q'$.

4. Phonon attenuation and renormalized coupling constant

Imaginary part of the pole of $D(q, \omega)$ gives the phonon attenuation which can be measured by the quantity

$$\delta_1(q) = \frac{F\frac{\pi|\omega|}{2p_0q}\omega_q^{02}}{(1-F/2+F\,q^2/16p_0^2)^2+F^2\left(\frac{\pi|\omega|}{2p_0q}\right)^2} \qquad (4.1)$$

An approximate expression for relative attenuation is

$$\frac{\delta_1(q)}{\omega} = \frac{\pi|\omega|}{2p_0q}\omega_q^0 F(1+5F/4). \qquad (4.2)$$

The residue at the pole of $D(q, \omega)$ defines the new coupling constant

$$2\omega_q|\alpha(q)|^2 = \frac{\frac{4\pi e^2}{q^2}\omega_q^2\left(1-\frac{p_0e^2}{\pi q^2}+\frac{e^2}{8\pi p_0}+i\frac{e^2|\omega|}{2q^3}\right)}{\left(1-\frac{p_0e^2}{\pi q^2}+\frac{e^2}{8\pi p_0}\right)^2+\left(\frac{e^2|\omega|}{2q^3}\right)^2} \qquad (4.3)$$

which can be approximated to the form :

$$2\omega q|\alpha(q)|^2 = \frac{2\pi^2}{p_0}F\omega_q^{02}(1+F). \qquad (4.4)$$

The expression for renormalized phonon propagator therefore takes the following form :

$$D(q, \omega) = \frac{2\omega_q|\alpha(q)|^2}{\omega^2-\omega_q^2-i\delta_1(q)} + \frac{2\omega_q|\alpha(q)|^2}{\omega_q^2} + \frac{i\,\delta_1(q)2\omega_q|\alpha(q)|^2}{\omega_q^2(\omega^2-\omega_q^2-i\delta_1(q)} \quad \ldots \quad (4.5)$$

In this expression it is to be noted that since the relative attenuation $\delta_1(q)/\omega_q$ is less than unity, the last term may be neglected in comparison with the other two terms.

5. Electron energy spectrum

As shown by Galitskii & Migdal (1958) the electron spectrum is determined by the equation

$$\epsilon_p^0-(\epsilon_p-i\Gamma_p)-\tilde{\Sigma}(p, \epsilon_p-i\Gamma_p) = 0 \qquad \ldots \quad (5.1)$$

where $\bar{\Sigma}(p, \epsilon)$ is the analytic continuation of $\Sigma(p, \epsilon)$ for $\epsilon > \mu$, the chemical potential. For $\Gamma_{p/\epsilon p} \ll 1$, one will obtain approximately

$$\epsilon_p{}^0 - \epsilon_p - \Sigma_0(p, \epsilon_p) = 0 \qquad \ldots \quad (5.2)$$

$$\Gamma_p = \Sigma_I(p, \epsilon_p) / \left\{ 1 + \left(\frac{\partial \Sigma_0}{\partial \epsilon} \right)_{\epsilon = \epsilon_p} \right\} \qquad \ldots \quad (5.3)$$

where Σ_0 and Σ_I are real and imaginary parts of electron self-energy.

To find out the expressions for Σ_0 and Σ_I let us first perform the substitution

$$\int \frac{d^3q}{(2\pi)^3} = \int d\epsilon_p \int \frac{d\eta}{4\pi} \cdot \frac{mp}{2\pi^2} \simeq \frac{mp_0}{2\pi^2} \int d\epsilon \int \frac{d\eta}{4\pi}$$

where

$$\eta \quad \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \simeq \frac{p}{p_0}, \quad \boldsymbol{p} - |\boldsymbol{q} - \boldsymbol{q}'|$$

Now let us denote the unperturbed density of states in energy at the Fermi surface by

$$N(0) = \frac{mp_0}{2\pi^2}$$

so that the electron self-energy integral takes the form :

$$\Sigma(p, \epsilon) = \frac{N(0)}{i} \int \frac{d\omega'}{2\pi} \int \frac{d\eta'}{4\pi} \int_0^w d\epsilon \, \frac{1}{Z - \epsilon} D(q, \omega'). \qquad \ldots \quad (5\,4)$$

To evaluate the ϵ-integral appearing in the above equation, we shall adopt the following simple procedure. It is a fact that the electrons lying far away from the Fermi surface contribute very little to the above integral and the major contribution results from the electron states with energies $\epsilon \leqslant \omega$, the Debye energy, about the Fermi surface. Therefore, if the upper limit of the integral is extended to infinity no perceptible error will creep in the final result Thus, extending the cut-off to infinity and maintaining the particle-hole symmetry by carrying out the ϵ-integration symmetrically about the Fermi surface $\epsilon = 0$, we get by closing the ϵ-contour in the upper half-plane

$$\Sigma(p, \epsilon) = N(0) \int d\omega' \int \frac{d\eta'}{4\pi} \left\{ \frac{2\omega_q |\alpha(q)|^2}{\omega^2 - \omega_q{}^2 - i\delta_1(q)} + \frac{2\omega_q |\alpha(q)|^2}{\omega_q{}^2} \right\} \qquad \ldots \quad (5.5)$$

with

$$q = p_0(\eta - \eta').$$

Performing the integration, we obtain the following expressions for Σ_0 and Σ_I

$$\Sigma_I(p, \epsilon) = -\frac{2\omega_q |\alpha(q)|^2 N(0)}{2}\left\{\frac{\pi}{\omega}\operatorname{sgn}\omega_q + \frac{1}{\omega_q^2}\right\} \quad \text{for } |\omega_q| > \omega$$

$$= -\frac{2\omega_q |\alpha(q)|^2 N(0)}{2\omega_q^2} \quad \text{for } |\omega_q| < \omega \qquad \ldots (5.6)$$

and

$$\Sigma_0(p, \epsilon) = -\frac{2\omega_q |\alpha(q)|^2 N(0)}{2\omega}\left\{\ln\left|\frac{\omega_q+\omega}{\omega_q-\omega}\right| + \frac{1}{\omega_q^2}\right\} \qquad \ldots (5.7)$$

Substituting this expression in (5.1) we get the electron energy spectrum of an electron-phonon-Coulomb system.

6. Graphical estimation of spectral weight function

Spectral weight function is in a certain sense, a generalization of the density matrix obtained after an integration over the frequencies Its general form can be presented as

$$A(\epsilon_p, \omega_q) = \frac{1}{\pi}\cdot\frac{|\Sigma_I(p, \epsilon)|}{[\omega_q - \epsilon_p - \Sigma_0(p, \epsilon)]^2 + [\Sigma_I(p, \epsilon)]^2} \qquad \ldots (6.1)$$

Analysis of such a function is important in the sense that it enables one to estimate the nature of the spectra of quasiparticles of a system. To estimate the quasiparticle spectra of an electron-phonon-Coulomb system we neglect the ω_q^{-2} terms in the expressions (5.6) and (5.7) and substituting these eqns. in (6.1), we get

$$\omega A(\epsilon_q, \omega_q) = \frac{\alpha_r^2/2}{\left[x - \frac{\epsilon_p}{\omega} - \frac{\alpha_r^2}{2}\ln\left|\frac{x+1}{x-1}\right|\right]^2 + \pi^2\alpha_r^4/4} \quad \text{for } |x| > |1 \qquad \ldots (6.2)$$

$$= \delta\left(x - \frac{\epsilon_p}{\omega} + \frac{\alpha_r^2}{2}\ln\left|\frac{x+1}{x-1}\right|\right) \quad \text{for } |x| < |1 \qquad \ldots (6.3)$$

where

$$\alpha_r^2 = \frac{2\omega_q |\alpha(q)|^2 N(0)}{\omega^2}, \quad x = \omega_q/\omega$$

Henceforth, we shall call α_r^2 the renormalized coupling constant. With the use of the eqns. (3.10) and (4.4) it will be found that for normal metals α_r^2 lies between 0 3 to 0.15. The corresponding unrenormalized range is $0.5 \geqslant \alpha_0^2 \geqslant 0.25$.

$\omega A(\epsilon_p, \omega_q)$ has been plotted against ω_q/ω for various values of ϵ_p. It is evident from above equations a quasiparticle peak will occur at $\epsilon_p/(1+\alpha_r^2)$ for $0 < \epsilon_p \ll \omega$

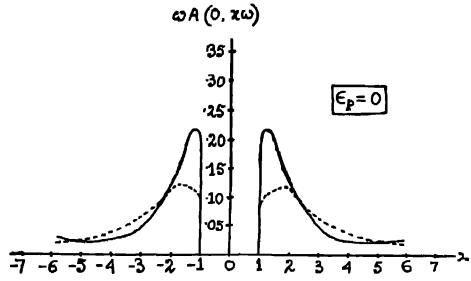

Figure Spectral weight function for electron-phonon (EP, dotted curve) and election-phonon-Coulomb (EPC full curve) systems. Renormalized coupling costant = 0 25 and unrenormalized coupling constant = 0.41

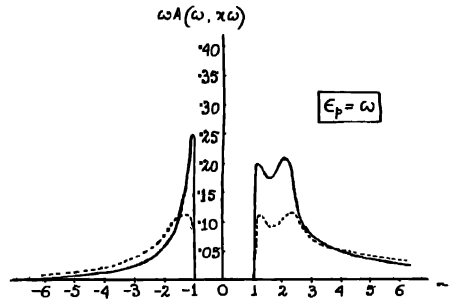

Figure 2. Spectral weight function for EP and EPC systems drawn for $c_p = \omega$. Renormalized and unrenormalized constants are the same as in figure 1.

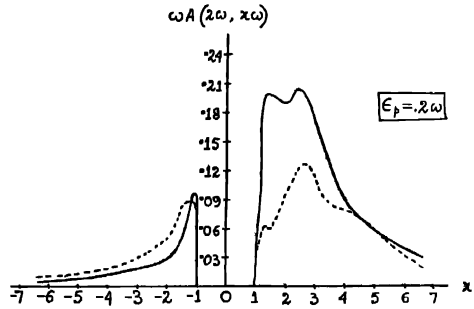

Figure 3 Spectral weight function for $c_p = 2\omega$. Coupling constants are the same as in figure 1.

and the maximum value of the spectral weight function in the continuum at $\epsilon_p = \omega(1+\alpha_r^2)^{\frac{1}{2}}$. With the increase of ϵ_p, quasiparticle peak moves towards ω, finally it approaches ω as ϵ_p tends to infinity. The quasiparticle peak at $\epsilon_p/(1+\alpha_r^2)$ may be regarded as corresponding to an electron immersed in an environment of virtual phonon cloud.

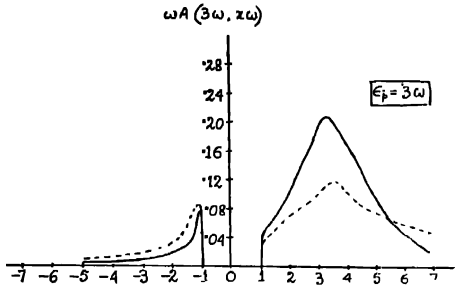

Figure 4. Spectral weight function for $\epsilon_p = 3\omega$, renormalized and unrenormalized coupling constants remaining the same as in the previous cases

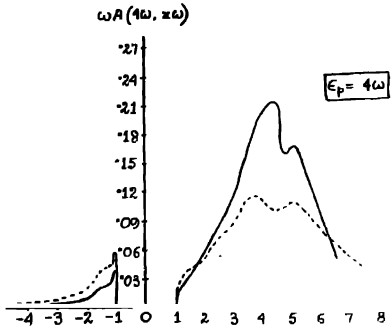

Figure 5 Spectral weight function for $\epsilon_p = 4\omega$. Values of the renormalized and unrenormalized coupling constants are 0.25 and 0.41 respectively.

It may be noted in this connection that the process of extending the ϵ-integration symmetrically to infinity, although capable of explaining essential features, has the chief drawback that it fails to throw any light on the mechanism of multiple -phonon emission.

In figure 6, $|\Sigma_0|/\omega$ has been plotted for various values of x assuming $\alpha_r^2 = 0.25$. The corresponding curve for an electron-phonon system without Coulomb interaction has been represented by dotted lines. The result one obtains is that the presence of Coulomb interaction decreases the coupling constant.

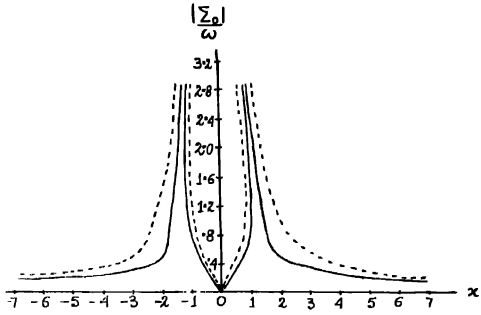

Figure 6. Variation of $|\Sigma_0|/\omega$ with ω_q/ω for EP (dotted) and EPC (full) system. Renormalized coupling constant = 0.25. Unrenormalized coupling constant = 0.41.

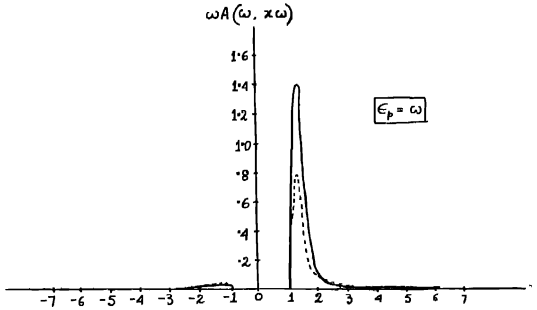

Figure 7. Spectral weight function for EP (broken curve) and EPC (full curve) systems α(un-renormalized = 0.12, α(renormalized) = 0.07 and $\epsilon_p = \omega$.

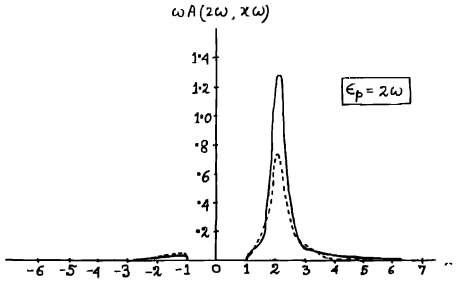

Figure 8. Spectral weight function for $\epsilon_p = 2\omega$. α(unrenormalized) = 0.12 and α(renormalised) = 0.07

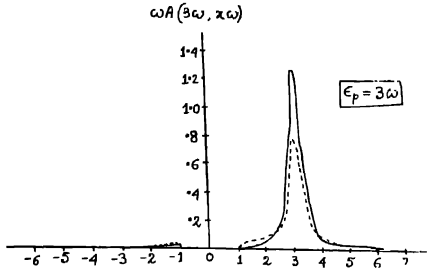

Figure 9. Spectral weight function for $\epsilon_p = 3\omega$. α(unrenormalised) = 0.12 and α(renormalized) = 0.07.

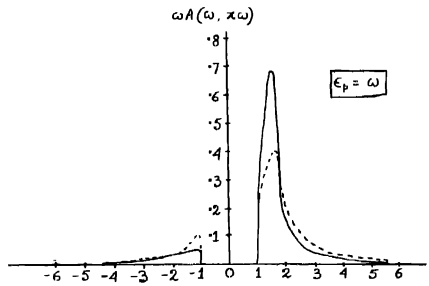

Figure 10. Spectral weight function for $\epsilon_p = \omega$. α(renormalized) = 0.5 and α(unrenormalized) = 0.25.

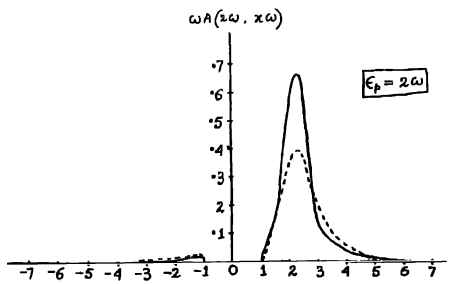

Figure 11. Spectral weight function for $\epsilon_p = 2\omega$, α(renormalized) = 0.15, and α(unrenormalized) = 0.25.

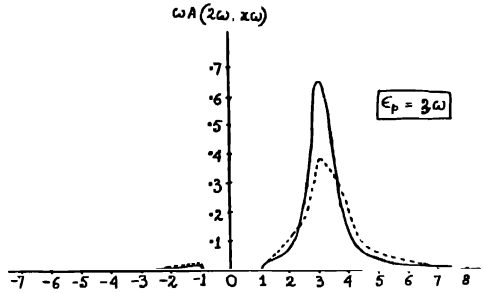

Figure 12. Spectral weight function for EP and EPC systems. α(renormalized) = 0.15. α(unrenormalized) = 0.25 and $c_p = 3\omega$.

Considering the value of the unrenormalized coupling constant corresponding to a particular value of the renormalized coupling constant, curves (represented by dotted lines) are drawn taking x as the abscissae and ωA as the ordinate on the same figure where ωA for an electron phonon Coulomb system has been plotted against x. It may be noted that the position of the quasiparticle peak in the two systems are different.

7. Electrons and Phonons near the Fermi surface

It is now an established fact that a quasiparticle on the Fermi Surface has an infinite lifetime. On this basis, we can neglect, to a reasonable approximation, the imaginary part of the self energy of a quasiparticle lying very near the Fermi surface. Thus writing

$$\Pi = \frac{p_0}{4\pi^2}\left(1-\frac{q^2}{8{p_0}^2}\right) \qquad \ldots (7.1)$$

$$\Sigma = -\frac{N(0)2\omega_q|\alpha(q)|^2}{2\omega}\ln\left|\frac{\omega_q+\omega}{\omega_q-\omega}\right| \qquad \ldots (7.2)$$

Due to the smallness we have neglected the second term of the eqn. (5.7)

It is to be noted from above two expressions, the ratio

$$\frac{\Sigma}{\Pi} = -\frac{4\pi^4 N(0)}{\omega {p_0}^2} F\ {\omega_q}^{02}(1+F)(1-q^2/8{p_0}^2)\ln\left|\frac{\omega_q+\omega}{\omega_q-\omega}\right| \qquad \ldots (7.3)$$

$$= -\frac{4\pi^4 N(0)}{\omega p_0}\ {c_0}^2 q^4(1+F)(1-q^2/8{p_0}^2)\ln\left|\frac{\omega_q+\omega}{\omega_q-\omega}\right| \qquad \ldots (7.4)$$

is approximately

$$\sim .6\omega\ \ln\left|\frac{\omega_q+\omega}{\omega_q-\omega}\right|$$

It is evident from (7.4) that for all possible realistic values of ω and ω_q, the ratio is a very small quantity in view of the very small factor $.6\omega$ appearing in the above expression. This means that phonon self-energy is much greater than the electron self-energy.

8. Inclusion of ionick motion

To include the effects due to ionic motion, we consider the Bardeen-Pines (1955) Hamiltonian

$$H = \sum_p \epsilon(p) c_p^+ c_p + \sum_{q,\lambda} \omega_{q\lambda}{}^0 a^+_{q\lambda} a_{q\lambda} + \sum_k q_k v_k{}^i \rho_{-k} + \tfrac{1}{2} \sum_k \frac{4\pi e^2}{k^2} \rho_{-k} \rho_k \quad \ldots (8.1)$$

where

$$\rho_k = \sum_{k,\sigma} c^+_{k+p,\sigma} c_{p,\sigma} ; \quad v_k{}^i = - \left(\frac{4\pi z e^2 i}{k} \right) (N/M)^{\frac{1}{2}}$$

The polarization propagator

$$D^{-1}(q, \omega) = [V_{\text{eff}}(q, \omega) + D_0(q, \omega)]^{-1} - \Pi \quad \ldots (8.2)$$

where

$$V_{\text{eff}}(q, \omega) = \frac{v_p{}^i}{\epsilon_{RPA}(q, \omega)} \quad \ldots (8.3)$$

becomes after simplifications as done previously

$$D(q, \omega) = \frac{2\omega_q |\alpha(q)|^2}{\omega^2 - \omega_q{}^2 - i\delta_1{}^i(q)} + \frac{\frac{v_q{}^i}{\epsilon} 2\omega_q |\alpha(q)|^2}{\xi \omega_q{}^{02} + \frac{v_q{}^i}{\epsilon} \omega_q{}^2}$$

$$+ \frac{i \frac{v_q{}^i}{\epsilon} \delta_1{}^i(q) 2\omega_q |\alpha(q)|^2}{(\omega^2 - \omega_q{}^2 - i\delta_1{}^i(q)) \left(\xi \omega_q{}^{02} + \frac{v_q{}^i}{\epsilon} \omega_q{}^2 \right)} \quad \ldots (8.4)$$

where the new renormalized coupling constant is

$$2\omega_q |\alpha(q)|^2 = \frac{\xi \omega_q{}^{02} + \frac{v_q{}^i}{\epsilon} \omega_q{}^2}{1 - \frac{v_q{}^i}{\epsilon} \frac{p_0}{4\pi^2} g(q/2p_0) - i \frac{v_q{}^i}{\epsilon} \frac{p_0}{4\pi} \frac{|\omega|}{2p_0 q}} \quad \ldots (8.5)$$

and the phonon attenuation is

$$\delta_1{}^i(q) = \frac{\xi \omega_q{}^{02} \frac{|\omega|}{8\pi q}}{1 - \frac{v_q{}^i}{\epsilon} \frac{p_0}{2\pi^2} g(q/2p_0)} \quad \ldots (8.6)$$

while the renormalized phonon frequency is found to be

$$\omega_q^2 = \omega_q^{02} \frac{\xi \frac{p_0}{4\pi^2} g(q/2p_0) \left\{1 - \frac{v_q^i}{\epsilon} \frac{p_0}{4\pi^2} g(q/2p_0)\right\} - \frac{v_q^i}{\epsilon} \xi \left\{\frac{\pi p_0}{4\pi^2} \cdot \frac{|\omega|}{2p_0 q}\right\}}{\left\{1 - \frac{v_q^i}{c} \frac{p_0}{4\pi^2} g(q/2p_0)\right\}^2 + \left\{\frac{v_q^i}{\epsilon} \cdot \frac{\pi p_0}{4\pi^2} \cdot \frac{|\omega|}{2p_0 q}\right\}^2} \quad \ldots (8.7)$$

In all these expressions ξ stands for

$$\xi = (4\pi e^2/q^2) - (v^i{}_q/\epsilon_{\mathrm{RPA}}(q, \omega))$$

$\epsilon_{\mathrm{RPA}}(q, \omega)$ being the electronic dielectric constant in the Random Phase Approximation.

In the expression (8 4), we neglect the last term due to smallness. We get, therefore,

$$\Sigma_I(p, \epsilon) = \frac{2\omega_q |\alpha(q)|^2 N(0)}{2} \left\{\frac{\pi}{\omega} \operatorname{sgn} \omega_q + \frac{v^i{}_q/\epsilon}{\xi\omega_q{}^{02} + \frac{v^i{}_q}{\epsilon}\omega^i{}_q}\right\} \text{ for } |\omega_q| > \omega$$

$$= -\frac{2\omega_q |\alpha(q)|^2 N(0) v^i{}_q/\epsilon}{2\left(\xi\omega_0{}^{02} + \frac{v_q^i}{c}\omega_q{}^2\right)} \quad \text{for } |\omega_q| < \omega \quad \ldots (8.8)$$

$$\Sigma_0(p, \epsilon) = -\frac{2\omega_q |\alpha(q)|^2 N(0)}{2\omega} \left\{\ln\left|\frac{\omega_q + \omega}{\omega_q - \omega}\right| + \frac{v^i{}_q/\epsilon}{\xi\omega_q{}^{02} + \frac{v^i{}_q}{\epsilon}\omega^3{}_q}\right\} \quad \ldots (8\,9)$$

These equations may be compared with the eqns. (5.6) and (5 7).

9. Summary

Quasiparticle spectra of an electron-phonon-Coloumb (EPC) system have been examined with the special emphasis on the explicit derivation of electron self-energy and on the graphical estimation of the spectral weight function for different values of phonon-emission threshold. Several curves are drawn for different values of electron energy, taking a specific value of the coupling constant. For the sake of comparison, the spectral weight function for an electron-phonon (EP) system has been plotted taking the corresponding unrenormalized value of the coupling constant. A special feature is to be noted from graphs; the curve for an EPC system lies above the curve for an EP System in the region $x < 1$ and below in the region $x < 1$. In section 8, the results have been extended to derive expression for electron's self-energy for an electron-phonon-Coulomb-ion (EPCI) System

The whole analysis, however, is based on Migdal's theorem which according to Engelsberg & Schrieffer (1963) has exceptions in certain special cases, (Section 2) In addition to this, the analysis works on Debye model of phonon spectra which is an idealized model. In spite of this, one may safely hope the results to agree closely with reality.

In Section 7, an interesting result has been obtained on the basis of very simple arguments. This may be stated as a theorem : the phonon self-energy is much greater than the electron self-energy in the neighbourhood of the Fermi Surface.

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